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EDGE-DISJOINT HOMOTOPIC PATHS IN STRAIGHT-LINE PLANAR GRAPHS*

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Abstract. Let G be a planar graph, embedded without crossings in the euclidean plane $\mathbb{R}^2$, and let $I_1, \cdots, I_p$ be some of its faces (including the unbounded face), considered as open sets. Suppose there exist (straight) line segments $L_1, \cdots, L_t$ in $\mathbb{R}^2$ so that $G \cup I_1 \cup \cdots \cup I_p = L_1 \cup \cdots \cup L_t \cup I_1 \cup \cdots \cup I_p$ and so that each L_i has its end points in $I_1 \cup \cdots \cup I_p$. Let $C_1, \cdots, C_k$ be curves in $\mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ with end points in vertices of G. Conditions are described under which there exist pairwise edge-disjoint paths $P_1, \cdots, P_k$ in G so that P_i is homotopic to C_i in $\mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$, for $i = 1, \cdots, k$. This extends results of Kaufmann and Mehlhorn for graphs derived from the rectangular grid.

Key words. edge-disjoint, paths, homotopic, packing, planar

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1. Introduction and statement of the theorem. Let $G = (V, E)$ be a planar graph, embedded without crossing edges in the euclidean plane $\mathbb{R}^2$. We identify G with its image in $\mathbb{R}^2$. Let $I_1, \cdots, I_p$ be some of its faces, including the unbounded face, called the *black holes*. (We consider faces as *open* sets.) Moreover, let paths $C_1, \cdots, C_k$ be given with end points in V, not intersecting any black hole. (That is, for each i, C_i is a continuous function $[0, 1] \to \mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ with $C(0), C(1) \in V$.)

Motivated by the automatic design of integrated circuits, Mehlhorn posed the following question:

(1) Under which conditions do there exist pairwise edge-disjoint paths $P_1, \cdots, P_k$ in G so that P_i is homotopic to C_i in the space $\mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ (for $i = 1, \cdots, k$)?

Here a *path* in G is a continuous function P: $[0, 1] \to G$ with $P(0)$, $P(1) \in V$. Paths $P_1, \cdots, P_k$ are *pairwise edge-disjoint* if the following holds: if $P_i(x) = P_j(y) \notin V$ then $x = y$ and $i = j$. (In particular, if $P_1, \cdots, P_k$ are pairwise edge-disjoint, then each P_i does not pass the same edge more than once.) Two paths P, C: $[0, 1] \to \mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ are *homotopic* (*in* $\mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$), denoted by $P \sim C$, if there exists a continuous function Φ: $[0, 1] \times [0, 1] \to \mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ so that for all $x \in [0, 1]$: $\Phi(x, 0) = P(x)$, $\Phi(x, 1) = C(x)$, $\Phi(0, x) = P(0)$, $\Phi(1, x) = P(1)$. (In particular, $P(0) = C(0)$ and $P(1) = C(1)$.)

Mehlhorn proposed to study question (1) with the help of the following "cuts." A (*homotopic*) *cut* is a continuous function D: $[0, 1] \to \mathbb{R}^2 \backslash (V \cup I_1 \cup \cdots \cup I_p)$ so that $D(0)$ and $D(1)$ belong to the boundary of $I_1 \cup \cdots \cup I_p$ and so that $|D^{-1}(G)|$ is finite. The *cut condition* (*for* $G; I_1, \cdots, I_p; C_1, \cdots, C_k$) is:

(2) (*cut condition*) for each cut D: $\operatorname{cr}(G,D) \geqq \sum_{i=1}^{k} \operatorname{mincr}(C_i, D)$.

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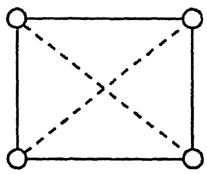
Here we use the following notation for curves $C, D: [0, 1] \to \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$:

$$\operatorname{cr}(G,D) := |\{y \in [0,1] \mid D(y) \in G\}|,$$

$$(3) \qquad \operatorname{cr}(C,D) := |\{(x,y) \in [0,1] \times [0,1] \mid C(x) = D(y)\}|,$$

$$\operatorname{mincr}(C,D) := \min\{\operatorname{cr}(\tilde{C},\tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D \text{ in } \mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)\}.$$

Clearly, the cut condition is a necessary condition for a positive answer to question (1). It is generally not sufficient, not even for quite simple situations. For example, take $k = 2$, $p = 1$, and consider



I_1,

where the straight lines stand for edges of G and where the interrupted lines stand for curves C_1 and C_2.

It turned out that one additional condition, the so-called *parity condition*, can be helpful (cf. § 2 below):

(4) (*parity condition*) for each cut D: $\operatorname{cr}(G, D) \equiv \sum_{i=1}^{k} \operatorname{mincr}(C_i, D) \pmod 2$.

Let us now state our theorem. We say that $G; I_1, \cdots, I_p; C_1, \cdots, C_k$ is in the *straight-line case* if

(5) there are line segments $L_1, \cdots, L_t$ in $\mathbb{R}^2$ so that $G \cup I_1 \cup \cdots \cup I_p = L_1 \cup \cdots \cup L_t \cup I_1 \cup \cdots \cup I_p$ and so that each L_j has its end points in $I_1 \cup \cdots \cup I_{p'}$

and

(6) if the aperture at vertex v of G is larger than 180°, then the number of times v occurs as end point of the curves C_i is not larger than the number of edges terminating at v.

Here the *aperture* at vertex v of G is the largest angle that can be made at v so that none of the black holes adjacent to v intersect the interior of the angle. (More formally, let $\rho > 0$ be so that the circle K of radius ρ and centre v does not contain any other vertex of G in its interior and does not intersect any edge except for those adjacent to v. Let $K \setminus (I_1 \cup \cdots \cup I_p)$ have components $K_1, \cdots, K_h$, making angles $\varphi_1, \cdots, \varphi_h$. Then the aperture at v is equal to $\max\{\varphi_1, \cdots, \varphi_h\}$.) Edge $e = \{(1 - \lambda)u + \lambda v \mid 0 < \lambda < 1\}$ of G is said to *terminate* at v if for some $\mu > 1$ the set $\{(1 - \lambda)u + \lambda v \mid 1 < \lambda < \mu\}$ is contained in $I_1 \cup \cdots \cup I_p$.

THEOREM. *If we are in the straight-line case and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

As an illustration, Fig. 1 gives an example of the straight-line case (where the shaded faces, together with the unbounded face, are the black holes, and where the interrupted curves stand for the paths C_i).

The theorem generalizes a result of Kaufmann and Mehlhorn [2] for graphs derived from the rectangular grid in the following way. G is a finite subgraph of the rectangular grid. (That is, V is a finite subset of $\mathbb{Z}^2$ and each edge is a line segment of length 1.) $I_1, \cdots, I_p$ are exactly those faces of G that are not bounded by exactly four edges of G.

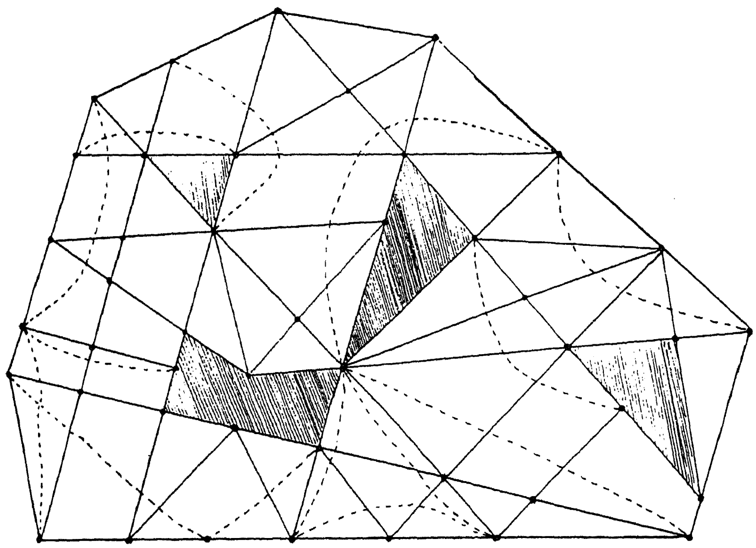


FIG. 1

Moreover, for each vertex v it is required that $\deg(v) + r(v) \leqq 4$, where $\deg(v)$ denotes the degree of v in G, and

$$r(v) := |\{i = 1, \cdots, k \mid C_i(0) = v\}| + |\{i = 1, \cdots, k \mid C_i(1) = v\}|.$$

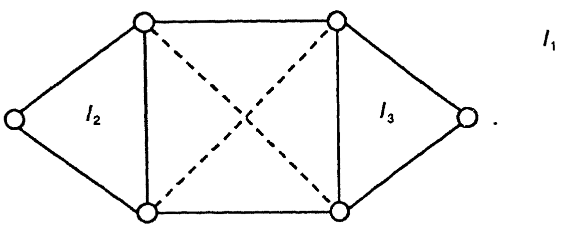
COROLLARY (Kaufmann and Mehlhorn). *If the conditions given in the previous paragraph are satisfied and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

In fact, Kaufmann and Mehlhorn found a linear-time algorithm to find these paths, if they exist.

In § 4 we give a proof of our theorem. We make use of a lemma to be proved in § 3 (showing that in the straight-line case we may restrict the cut condition to (almost) straight cuts (analogous to the idea of "1-bend cuts" in [2])), and of results of [4] to be reviewed in § 2.

2. Review of preliminary results. In this section we return to the general case of a planar graph $G = (V, E)$ embedded without crossing edges in the Euclidean plane $\mathbb{R}^2$, with black holes $I_1, \cdots, I_p$ (including the unbounded face) and curves $C_1, \cdots, C_k$. Let each C_i have its end points in vertices on the boundary of $I_1 \cup \cdots \cup I_p$.

It was shown by Okamura and Seymour [3] that if $p = 1$ the cut condition together with the parity condition imply the existence of paths as in (1). (Note that for $p = 1$ two paths P, P' are homotopic if and only if $P(0) = P'(0)$ and $P(1) = P'(1)$.) This was extended by van Hoesel and Schrijver [1] to $p = 2$. It cannot be extended to higher p, as is shown for $p = 3$ by:



However, it was shown in [4] that, for arbitrary p, the cut condition is equivalent to the existence of a "fractional" packing of paths as required, i.e., to the existence of paths $P_1^1, \cdots, P_1^{t_1}, P_2^1, \cdots, P_k^1, \cdots, P_k^{t_k}$ and rationals $\lambda_1^1, \cdots, \lambda_1^{t_1}, \lambda_2^1, \cdots, \lambda_k^1, \cdots, \lambda_k^{t_k} > 0$ such that:

(7)
$$\text{(i)}\quad P_i^j \sim C_i \qquad (i=1,\cdots,k; j=1,\cdots,t_i),$$
$$\text{(ii)}\quad \sum_{j=1}^{t_i} \lambda_i^j = 1 \qquad (i=1,\cdots,k),$$
$$\text{(iii)}\quad \sum_{i=1}^{k}\sum_{j=1}^{t_i} \lambda_i^j \chi^{P_i^j}(e) \leqq 1 \qquad (e \in E).$$

Here $\chi^P(e)$ denotes the number of times path P passes edge e.

Another result from [4] to be used below was derived with the theory of simplicial approximations. Let C, D: $[0, 1] \rightarrow \mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p)$ be continuous. Let $C(0)$, $C(1)$, $D(0)$, and $D(1)$ be on the boundary of $I_1 \cup \cdots \cup I_p$, with $\{C(0), C(1)\} \cap \{D(0), D(1)\} = \emptyset$. Let

(8)
$$X := \{(y,z) \in [0,1] \times [0,1] \mid C(y) = D(z)\}$$

be finite, where each (y, z) in X gives a crossing of C and D. For $y, y' \in [0, 1]$ let $C|_y^{y'}$ denote the path from $C(y)$ to $C(y')$ given by:

(9)
$$(C|_y^{y'})(\lambda) := C((1-\lambda)y + \lambda y') \quad \text{for } \lambda \in [0,1];$$

similarly for D. Define for $(y, z), (y', z') \in X$:

(10)
$$(y,z) \approx (y',z') \Leftrightarrow (C|_y^{y'}) \approx (D|_z^{z'}) \quad \text{in } \mathbb{R}^2 \backslash (I_1 \cup \cdots \cup I_p).$$

We call the classes of the equivalence relation $\approx$ the *classes of intersections* of C and D. Such a class is called *odd* if it contains an odd number of elements. Let odd (C, D) denote the number of odd classes of X. Then

(11)
$$\text{mincr}\,(C,D) = \text{odd}\,(C,D).$$

3. A lemma on straight cuts. We call a cut D: $[0, 1] \rightarrow \mathbb{R}^2 \backslash (V \cup I_1 \cup \cdots \cup I_p)$ a *straight cut* if

(12)
either (i) D is linear,
or (ii) the line segment connecting $D(0)$ and $D(1)$ is contained in G, the functions $D|[0, \frac{1}{2}]$ and $D|[\frac{1}{2}, 1]$ are linear, there is no vertex of G contained in the interior of the triangle $D(0)D(\frac{1}{2})D(1)$, and no edge is intersected more than once by D.

In (ii) we might think of D as being very close to the line segment connecting $D(0)$ and $D(1)$. So a straight cut is determined by its end points, in case (12) (ii) up to "slight" homotopic shifts, which, however, do not change the number of intersections with G.

LEMMA. *In the straight-line case, the cut condition holds if and only if* $\text{cr}\,(G, D) \geqq \sum_{i=1}^{k} \text{mincr}\,(C_i, D)$ *for each straight cut* D.

Proof. Necessity being trivial, we show sufficiency. Let the cut inequality be satisfied by each straight cut. Suppose there exists a cut D: $[0, 1] \rightarrow \mathbb{R}^2 \backslash (V \cup I_1 \cup \cdots \cup I_p)$ so that

(13)
$$\text{cr}\,(G,D) < \sum_{i=1}^{k} \text{mincr}\,(C_i,D).$$

We choose D satisfying (13) so that $t := \mathrm{cr}(G, D)$ is as small as possible. The idea of the proof is to straighten out D as much as possible.

First observe that we may assume that if $D(1)$ is not on the line through the edge containing $D(0)$, then the line segment $\overline{D(0)D(1)}$ does not intersect V (this can be achieved by slightly shifting $D(0)$ along the edge containing $D(0)$). Moreover, we may assume that there exists an $\varepsilon > 0$ so that

(14)
(i) $D|[0, \varepsilon]$ is linear;
(ii) for all $\delta \in (0, \varepsilon]$: $D(\delta)$ does not belong to any line through any pair of vertices of G nor to any line through a pair of points consisting of a vertex of G and an intersection of D and G.

Let $\lambda_1, \cdots, \lambda_t$ be so that $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{t-1} < \lambda_t = 1$, with $D(\lambda_i) \in G$ for all i. Define

$$p_1 := D(\varepsilon), \qquad p_i := D(\lambda_i) \quad \text{for } i = 2, \cdots, t. \tag{15}$$

Finally, we may assume that $D|[\varepsilon, \lambda_2]$ and $D|[\lambda_{i-1}, \lambda_i]$ are linear functions ($i = 3, \cdots, t$) (since in the straight-line case each face not in $\{I_1, \cdots, I_p\}$ is convex).

Let $h(D)$ be the smallest index h with $2 \leqq h \leqq t - 1$ so that the angle between $\overline{p_{h-1}p_h}$ and $\overline{p_hp_{h+1}}$ is not 180°. If no such h exists, let $h(D) := t$. We may assume that we have chosen D so that (fixing $t = \mathrm{cr}(G, D)$) $h(D)$ is as large as possible. Let $h := h(D)$.

First consider the case $h < t$. Choose the largest $\lambda \in [0, 1]$ so that the triangle with vertices p_1, p_h, and $p_h + \lambda(p_{h+1} - p_h)$ does not intersect $I_1 \cup \cdots \cup I_p$. Let $p'_h := p_h + \lambda(p_{h+1} - p_h)$. Let D' be the piecewise linear function obtained from D by replacing parts $\overline{p_1p_h}$ and $\overline{p_hp'_h}$ of D by $\overline{p_1p'_h}$.

If $\lambda = 1$, then $p'_h = p_{h+1}$, and hence by (14)(ii) $\overline{p_1p'_h}$ does not intersect any vertex of G. So D' is a cut, with $\mathrm{cr}(G, D') = \mathrm{cr}(G, D)$ (by the conditions (5) and (6) for the straight-line case) and $D' \sim D$. As $h(D') > h(D)$ this contradicts the fact that we have chosen D so that $h(D)$ is as large as possible.

If $\lambda < 1$, then $\overline{p_1p'_h}$ intersects a vertex v of G, on the boundary of $I_1 \cup \cdots \cup I_p$. This vertex is unique by (14)(ii) and has aperture larger than 180°. Consider a circle K with center v, not containing any other vertex of G, and not intersecting any edge of G except for those adjacent to v. Let $K \backslash (I_1 \cup \cdots \cup I_p)$ have components $K_1, \cdots, K_h$. So each K_i is a cut. We may assume that K_1 intersects D' twice. So K_1 is a circular arc of angle larger than 180°. Use the notation A, B, C, E, F for the parts of D' and K_1 as indicated in Fig. 2. Let H denote the part of D from p'_h to p_t. As we have chosen D so that (13) is satisfied with $\mathrm{cr}(G, D)$ as small as possible, we have

$$\begin{aligned}
\mathrm{cr}(G,D) &= \mathrm{cr}(G,EBFH) = \mathrm{cr}(G,EA) + \mathrm{cr}(G,CFH) + \sum_{j=2}^{h} \mathrm{cr}(G,K_j) \\
&\quad + (\text{number of edges terminating at } v) \\
&\geqq \sum_{i=1}^{k} \mathrm{mincr}(C_i,EA) + \sum_{i=1}^{k} \mathrm{mincr}(C_i,CFH) + \sum_{j=2}^{h}\sum_{i=1}^{k} \mathrm{mincr}(C_i,K_j) \\
&\quad + \sum_{i=1}^{k} (\text{number of times } v \text{ is end point of } C_i) \geqq \sum_{i=1}^{k} \mathrm{mincr}(C_i,D)
\end{aligned} \tag{16}$$

(using (6)). This contradicts (13).

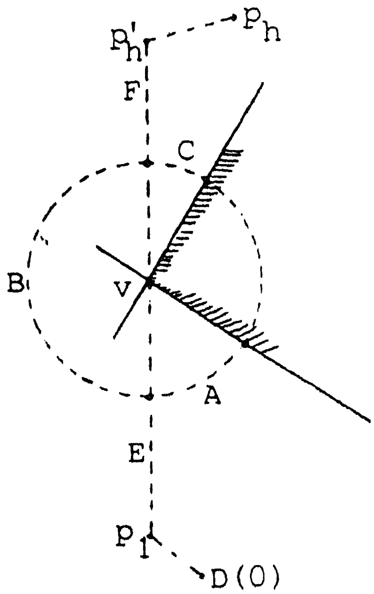


FIG. 2

As $h < t$ leads to a contradiction, we know $h = t$. If the line segment $\overline{D(0)D(1)}$ is not contained in G, then by our assumption this line segment forms a straight cut D', with $\text{cr}(G, D') = \text{cr}(G, D)$ and $D' \sim D$, whence

$$\text{cr}(G,D) = \text{cr}(G,D') \geqq \sum_{i=1}^{k} \text{mincr}(C_i, D') = \sum_{i=1}^{k} \text{mincr}(C_i, D), \tag{17}$$

contradicting (13). If $\overline{D(0)D(1)}$ is contained in G, then D itself forms a straight cut, contradicting (13). □

4. Proof of the theorem. We now prove our theorem.

THEOREM. *If we are in the straight-line case and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

Proof. The proof is by induction on the number of faces not in $\{I_1, \cdots, I_p\}$. If each face belongs to $\{I_1, \cdots, I_p\}$, then the theorem is trivially true. So assume that not all faces belong to $\{I_1, \cdots, I_p\}$.

I. We first consider those situations where the following holds:

(18) G has an edge e_0, connecting vertices u and w, both of degree 2, so that e_0 separates a face in $\{I_1, \cdots, I_p\}$ from a face not in $\{I_1, \cdots, I_p\}$ and so that one of the curves C_i connects u and w following e_0.

Without loss of generality, e_0 separates face I_1 from face $F \notin \{I_1, \cdots, I_p\}$, and C_1 connects u and w following e_0. Moreover, we may assume that none of $C_2, \cdots, C_k$ passes e_0 (we can make detours along the other edges of F). By the parity condition, there exist h, j so that C_h has an end point in u and C_j has an end point in w (possibly $h = j$).

Now let $I_{p+1} := F$. Clearly, $G; I_p, \cdots, I_p, I_{p+1}; C_1, \cdots, C_k$ is again in the straight-line case, in which the parity condition holds. We show

(19) the cut condition holds for $G; I_1, \cdots, I_{p+1}; C_1, \cdots, C_k$.

As the number of faces not in $\{I_1, \cdots, I_{p+1}\}$ is one less than in the original situation, (19) implies by induction that there exist pairwise edge-disjoint paths $P_1 \sim C_1, \cdots, P_k \sim C_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_{p+1})$. This implies $P_1 \sim C_1, \cdots, P_k \sim C_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ as required.

We prove (19). We will refer to $G; I_1, \cdots, I_{p+1}; C_1, \cdots, C_k$ as the *new structure*, and to $G; I_1, \cdots, I_p; C_1, \cdots, C_k$ as the *original structure*. For the new structure we use the notation mincr$'$ instead of mincr.

To show (19) by the lemma, it suffices to prove the cut inequality for straight cuts only. Let D be a straight cut in the new structure. If $D(0)$ and $D(1)$ belong to the boundary of $I_1 \cup \cdots \cup I_p$, then D is also a cut in the original structure, and the cut inequality follows (as $\operatorname{mincr}'(C_i, D) = \operatorname{mincr}(C_i, D)$ for each i). If both $D(0)$ and $D(1)$ belong to the boundary of $I_{p+1} = F$, then $\operatorname{mincr}'(C_i, D) = 0$ for each i (as F is convex), and the cut inequality follows. So we may assume that $D(0)$ belongs to the boundary of $I_1 \cup \cdots \cup I_p$ and $D(1)$ belongs to the boundary of F. We can extend D in $\bar{F}$ to a cut D' ending on e_0. Then D' is a cut in the original structure. Thus we have

$$(20) \qquad \operatorname{cr}(G,D) = \operatorname{cr}(G,D') - 1 \geqq \sum_{i=1}^{k} \operatorname{mincr}(C_i, D') - 1 = \sum_{i=1}^{k} \operatorname{mincr}'(C_i, D),$$

thus showing the cut inequality for D. This proves (19).

II. Now we consider the general case (i.e., we do not assume (18)). As not all faces belong to $\{I_1, \cdots, I_p\}$, there exists an edge, say e_0, separating a face I_h ($1 \leqq h \leqq p$) from a face F not in $\{I_1, \cdots, I_p\}$. We may assume $h = 1$. Without loss of generality, no path C_i intersects e_0 or F (we can make detours along the boundary of F). Extend G to a graph G' by adding two new vertices, say u and w, on e_0. Let e'_0 be the edge connecting u and w. Let C_{k+1} and C_{k+2} be two curves, each connecting u and w via e'_0. We consider two cases.

Case 1. The cut condition holds for $G'; I_1, \cdots, I_p; C_1, \cdots, C_k, C_{k+1}, C_{k+2}$. Now we can apply part I of this proof above, and paths $P_1, \cdots, P_k, P_{k+1}, P_{k+2}$ as required exist.

Case 2. The cut condition does not hold for $G'; I_1, \cdots, I_p; C_1, \cdots, C_k, C_{k+1}, C_{k+2}$. Since also in this new situation we are in the straight-line case, by the lemma there exists a straight cut D so that

$$(21) \qquad \operatorname{cr}(G', D) < \sum_{i=1}^{k+2} \operatorname{mincr}(C_i, D).$$

Since $\operatorname{mincr}(C_{k+1}, D) = \operatorname{mincr}(C_{k+2}, D) \leqq 1$ and since the parity condition holds for $G; I_1, \cdots, I_p; C_1, \cdots, C_k$ we know

$$(22) \qquad \operatorname{cr}(G, D) = \sum_{i=1}^{k} \operatorname{mincr}(C_i, D),$$

and $\operatorname{mincr}(C_{k+1}, D) = \operatorname{mincr}(C_{k+2}) = 1$. Hence D has one of its end points on e'_0.

As the cut condition holds for $G; I_1, \cdots, I_p; C_1, \cdots, C_k$, there exists a "fractional" packing of paths $P_1^1, \cdots, P_1^{t_1}, \cdots, P_k^1, \cdots, P_k^{t_k}$, with coefficients $\lambda_1^1, \cdots, \lambda_1^{t_1}, \cdots, \lambda_k^1, \cdots, \lambda_k^{t_k} > 0$, satisfying (7). By (22), at least one of the P_i^j, say P_1^1, passes edge e_0. So $P_1^1 = R_1 e'_0 R_2$ for certain paths R_1 and R_2.

We now show the following claim.

CLAIM. *For each straight cut D' (for G') we have*

$$(23) \quad \operatorname{mincr}(R_1, D') + \operatorname{mincr}(C_{k+1}, D') + \operatorname{mincr}(R_2, D') \leqq \operatorname{mincr}(C_1, D') + 2.$$

Proof of the claim. Since

$$(24)\qquad \operatorname{cr}(G,D)=\sum_{i=1}^{k}\operatorname{mincr}(C_i,D)\leqq\sum_{i=1}^{k}\sum_{j=1}^{t_i}\lambda_i^j\cdot\operatorname{cr}(P_i^j,D)\leqq\operatorname{cr}(G,D),$$

and since $\lambda_1^1>0$, we know that $\operatorname{cr}(P_1^1, D) = \operatorname{mincr}(C_1, D)$.

Without loss of generality, $(P_1^1|_0^{1/4})$ coincides with path R_1, $(P_1^1|_{1/4}^{3/4})$ with C_{k+1}, and $(P_1^1|_{3/4}^1)$ with R_2. Moreover, we may assume that $P_1^1(1/2) = D(0)$.

Let D' be any straight cut. To show (23) we may assume that D and D' intersect each other at most once, and that if D' intersects e_0', then D and D' do not intersect.

Let

$$(25)\qquad X:=\{(x,y)\in[0,1]\times[0,1]\mid P_1^1(x)=D'(y)\}.$$

Let $\approx$ be as in (10). So $\operatorname{mincr}(C_1, D')$ is equal to the number of odd classes of $\approx$. We show

(26) if $(x, y), (x', y'), (x'', y''), (x''', y''')\in X$ so that $(x, y)\approx(x', y')$, $(x'', y'')\approx(x''', y''')$, $x, x''\in(0, \frac{1}{2})$ and $x', x'''\in(\frac{1}{2}, 1)$, then D and D' intersect and $(x, y)\approx(x'', y'')$.

Indeed, as $(x, y)\approx(x', y')$, we know $(P_1^1|_x^{x'})\sim(D'|_y^{y'})$. So $(P_1^1|_x^{x'})(D'|_{y'}^{y})$ forms a homotopically trivial cycle K. Since $(P_1^1|_x^{x'})$ passes $D(0)$, D splits K into two homotopically trivial cycles. That is, there is a $\lambda\in(0, 1]$ so that

(27) either (i) $\exists z\in[x, x']$: $(P_1^1|_z^{1/2})(D|_0^\lambda)$ is a homotopically trivial cycle,
or (ii) $\exists z\in(y, y')$: $(P_1^1|_x^{1/2})(D|_{1/2}^{\lambda})(D'|_z^{y})$ is a homotopically trivial cycle.

Since $\operatorname{cr}(P_1^1, D) = \operatorname{mincr}(P_1^1, D)$, (27) (i) does not occur. So (27) (ii) applies. Hence

$$(28)\qquad (P_1^1|_x^{1/2})\sim(D'|_y^z)(D|_\lambda^{1/2}).$$

In particular, D and D' intersect, with $D(\lambda) = D'(z)$. We similarly derive from the fact that $(x'', y'')\approx(x''', y''')$ that

$$(29)\qquad (P_1^1|_{x''}^{1/2})\sim(D'|_{y''}^{z})(D|_\lambda^{1/2}).$$

Therefore,

$$(30)\qquad (P_1^1|_x^{x''})\sim(P_1^1|_x^{1/2})(P_1^1|_{1/2}^{x''})\sim(D'|_y^z)(D|_\lambda^{1/2})(D|_{1/2}^{\lambda})(D'|_z^{y''})\sim(D'|_y^{y''}).$$

So $(x, y)\approx(x'', y'')$. This shows (26).

Now $\operatorname{cr}(C_{k+1}, D')\leqq 1$. If $\operatorname{cr}(C_{k+1}, D') = 0$, then the above implies

$$(31)\qquad \operatorname{odd}(P_1^1,D')\geqq(\operatorname{odd}(R_1,D')-1)+(\operatorname{odd}(R_2,D')-1),$$

since by (26) all but at most one class of intersections of R_1 and D' is also a class of intersections of P_1^1 and D'. Similarly for R_2. Equation (31) implies (23).

If $\operatorname{cr}(C_{k+1}, D') = 1$, then D and D' do not intersect, by assumption. Hence, by (26), no class of intersections of P_1^1 and D' contains both (x, y) and (x', y') with $x\in(0, \frac{1}{2})$ and $x'\in(\frac{1}{2}, 1)$. Since $\operatorname{cr}(C_{k+1}, D') = 1$, there is only one element (x, y) in X with $x\in(\frac{1}{4}, \frac{3}{4})$. Except for the class of intersections of P_1^1 and D' containing this element, all other classes also form a class of intersections of R_1 and D' or of R_2 and D'. Hence

$$(32)\qquad \operatorname{odd}(P_1^1,D')\geqq\operatorname{odd}(R_1,D')+\operatorname{odd}(R_2,D')-1,$$

and (23) follows. □

We next show

(33) the cut condition holds for $G'; I_1, \cdots, I_p; R_1, R_2, C_2, \cdots, C_k, C_{k+1}$.

Suppose not. Since we are again in the straight-line case, by the lemma there exists a straight cut D' so that

$$\text{(34)} \qquad \operatorname{mincr}(R_1, D') + \operatorname{mincr}(R_2, D') + \sum_{i=2}^{k+1} \operatorname{mincr}(C_i, D') \geqq \operatorname{cr}(G, D') + 2,$$

using the fact that the parity condition holds also for $G'; I_1, \cdots, I_p; R_1, R_2, C_2, \cdots, C_{k+1}$. Since the cut condition does hold for $G'; I_1, \cdots, I_p; C_1, \cdots, C_k$, it follows that

$$\text{(35)} \qquad \operatorname{mincr}(R_1, D') + \operatorname{mincr}(R_2, D') + \operatorname{mincr}(C_{k+1}, D') > \operatorname{mincr}(C_1, D').$$

Hence

$$\text{(36)} \qquad \operatorname{cr}(P_1^1, D') = \operatorname{cr}(R_1, D') + \operatorname{cr}(R_2, D') + \operatorname{cr}(C_{k+1}, D') > \operatorname{mincr}(C_1, D').$$

Therefore,

$$\text{(37)} \qquad \begin{aligned} \operatorname{cr}(G, D') &\geqq \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_i^j \cdot \operatorname{cr}(P_i^j, D') > \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_i^j \cdot \operatorname{mincr}(C_i, D') \\ &= \sum_{i=1}^{k} \operatorname{mincr}(C_i, D'). \end{aligned}$$

However, (34) and (37) imply

$$\text{(38)} \qquad \begin{aligned} \operatorname{mincr}(R_1, D') + \operatorname{mincr}(R_2, D') + \sum_{i=2}^{k+1} \operatorname{mincr}(C_i, D') &\geqq \operatorname{cr}(G, D') + 2 \\ &> \sum_{i=1}^{k} \operatorname{mincr}(C_i, D') + 2, \end{aligned}$$

contradicting the claim.

So (33) holds, and hence by part I of this proof there exist pairwise edge-disjoint paths $Q_1' \sim R_1$, $Q_1'' \sim R_2$, $Q_2 \sim C_2, \cdots, Q_k \sim C_k$, $Q_{k+1} \sim C_{k+1}$. By sticking Q_1', Q_{k+1}, Q_1'' to one path, which is homotopic to C_1, we obtain paths as required. □

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